

Dynamics of periodic node states on a model of static networks with repeated-averaging rules

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We introduce a simple model of static networks, where nodes are located on a ring structure, and two accompanying dynamic rules of repeated averaging on periodic node states. We assume nodes can interact with neighbors, and will add long-range links randomly. The number of long-range links, E , controls structures of these networks, and we show that there exist many types of fixed points, when E is varied. When E is low, fixed points are mostly diverse states, in which node states are diversely populated; on the other hand, when E is high, fixed points tend to be dominated by converged states, in which node states converge to one value. Numerically, we observe properties of fixed points for various E 's, and also estimate points of the transition from diverse states to converged states for four different cases. This kind of simple network models will help us understand how diversities that we encounter in many systems of complex networks are sustained, even when mechanisms of averaging are at work, and when they break down if more long-range connections are added.

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I. INTRODUCTION

Complex networks have been a popular subject of statistical physics for more than a decade, because of their versatility and rich dynamic behavior[1, 2]. In addition, they might be the only way to explain some of the universal features of various systems that can be represented by networks. Even though structures of static networks can be a research subject of its own, studies on dynamic networks are also flourishing. Three approaches can be used on models of network dynamics, depending on the characteristics of systems in question: (i) node dynamics, focusing on dynamics of node states, while link structures are static, (ii) link dynamics, focusing on dynamics of links and their structures, while nodes are stateless, or have static states, and (iii) coevolving dynamics, where both links and nodes evolve, possibly influencing each other. Note that when some entities are assumed to be *static*, it means that the characteristic times for their changes are considerably longer than those of *dynamic* entities. One field that has attracted many physicists recently as an application of dynamic complex networks is a study of social systems[3, 4], where nodes are humans or groups of humans, and their relationships define links, forming social networks. Social entities influence each other, mostly in attracting ways, but one of the fundamental characteristics of our societies is the sustained diversities. How are they being sustained when social entities have tendencies to assimilate with each other? How and when do these diversities break down? These are the questions we try to answer using complex networks on problems of

opinion dynamics[5–11], cultural dynamics[12–14] and so on.

Here we introduce a model of static networks and investigate node dynamics on these networks using numerical simulations mostly, while we have already studied the same dynamic rule on dynamic networks[11, 15]. This model has nodes with periodic states with repeated-averaging rules[5, 16], and there are two types of links: short-range and long-range links. Long-range interactions are rapidly increasing in our societies, as we are recently seeing with ever-advancing technologies, and we will use the number of long-range links, E , as the control parameter of the model. In our model, E changes the network structure, and has an important role in determining what types of fixed points are reached from random initial conditions. When there are not many long-range links, diverse states prevail, but diversities will disappear eventually and more converged states will appear. As we vary E , we will find several types of fixed points, and estimate the likelihood of spontaneous emergence for given types of fixed points, using Monte-Carlo simulations. Network models with periodic nodes can be applied not only to social systems, but also to other domains as in synchronization problems in oscillator networks[17]. For example, there have been many works on dynamics of Kuramoto model[18, 19] on complex networks[20–26], and one dynamic rule in our model is similar to that of Kuramoto model.

The plan of this paper is as follows. In Sec. II, the network model and two dynamic rules are introduced, and dynamics of networks with a small number of nodes is examined. In Sec. III, numerical results are presented. We find different types of fixed points using Monte-Carlo simulations, and observe their properties as E varies. Finally, in Sec. IV, we conclude with discussions.

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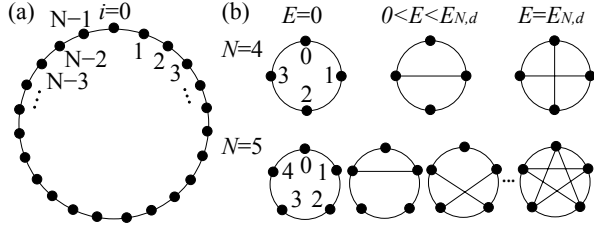


FIG. 1. (a) N nodes in an 1D periodic lattice with only nearest-neighbor connections ($d = 1$ here). (b) Examples of $N = 4$ and 5 when $d = 1$. We can add E long-range links, where $E_{N,d}$ is the maximum number of long-range connections for given N and d .

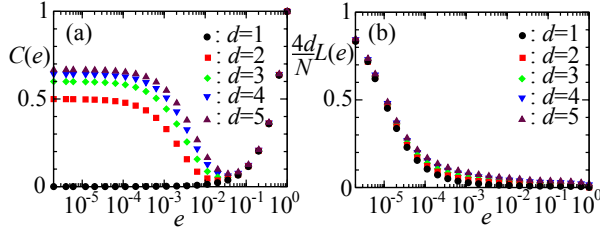


FIG. 2. (Color online) For $N = 1000$, we observe (a) the clustering coefficient, $C(e)$, and (b) the average shortest path length, $L(e)$, versus e for $1 \leq d \leq 5$ (averaged over 1000 random network configurations).

II. MODEL

We describe a network model with simple examples in Fig. 1. In this model, the network has N nodes, and they are placed in a ring structure, a 1D periodic lattice. Each node is connected to neighbors within distance d , assuming $N \gg d$; then, there are dN links, called *short-range* links. Now we add E distinct links randomly, not between already-connected neighbors, and these links are *long-range* links (or *shortcuts*). The total number of links becomes $dN + E$, and the average connectivity k is $2(d + E/N)$. If we introduce the normalized parameter $e \equiv E/E_{N,d}$, where $E_{N,d}$ is $N(N-1-2d)/2$, the maximum number of distinct long-range links, this becomes a normalized control parameter of the network ($0 \leq e \leq 1$) [27]. This model can be viewed as a modified version of the small-world network by Newman and Watts [28]. When $d = 0$, networks will have the same structural properties as random networks while E is the total number of links. When $e = 0$ and $d > 1$, the network is a regular network, and, as e increases, the network undergoes several structural phases including small-world networks for low e .

In Fig. 2, we observe the clustering coefficient $C(e)$ and the average shortest path length $L(e)$ as we vary e , for the cases for $N = 1000$ and $1 \leq d \leq 5$. When $e = 0$ and

$d \geq 1$, we can easily find $C(0)$ and $L(0)$ as below,

$$C(0) = \frac{3(d-1)}{2(2d-1)}, \quad (1)$$

$$L(0) = \left(\frac{q+1}{N-1} \right) \left(\frac{N-\delta}{2} + r \right) \simeq \frac{N}{4d}, \quad (2)$$

where q and r are the quotient and the remainder when dividing $(N-2+\delta)/2$ by d , and δ is defined as

$$\delta \equiv \begin{cases} 0 & (N: \text{even}), \\ 1 & (N: \text{odd}). \end{cases} \quad (3)$$

For $d > 1$, $C(e)$ does not change much until e reaches 2×10^{-4} ($E \simeq 100$), while $L(e)$ is reduced to about $L(0)/10$. As e increases further, $C(e)$ decreases sharply to almost 0 and starts to increase at $e \simeq 2 \times 10^{-2}$ ($E \simeq 10^4$), and finally reaches the fully-connected network at $e = 1$. On the other hand, $L(e)$ monotonically decreases until it reaches 1 at $e = 1$.

Now we introduce a state variable ϕ_i to each node i , and assume ϕ_i is periodic with the range of $0 \leq \phi_i < 1$. Then the state space of all N nodes, which can be represented by an n -tuple, $(\phi_0, \phi_1, \phi_2, \dots, \phi_{N-1})$, becomes an N -dimensional torus (N -torus). Now we impose a synchronous and deterministic dynamic rule with discrete time t ($t = 0, 1, 2, \dots$), and the rule can be represented by a map for each node i at each time step as below,

$$\phi_i \leftarrow \phi_i + \frac{\sigma}{k_i + 1} \sum_{j=0}^{N-1} K_{ij} \Delta(\phi_j - \phi_i) \pmod{1}, \quad (4)$$

where σ is a coupling constant ($0 < \sigma \leq 1$, but we will assume $\sigma = 1$ for simplicity here), K_{ij} is an element of the adjacency matrix (1 when there exists a link between i and j , and 0 otherwise), k_i is the degree of node i , and $\Delta(\phi_j - \phi_i)$ represents the difference between ϕ_j and ϕ_i . Then, the next value is the *average* of all states of neighbors of i and i itself when $\sigma = 1$. (If $\sigma < 1$, ϕ_i will move closer to the average, and, if $\sigma > 1$, which is not allowed here, the next value of ϕ_i will overshoot the average.)

If the state variable is not periodic, we can simply use $\Delta(x) = x$. But the state variable is periodic here, and the definition of averaging can become ambiguous. For example, if we have two values, 0.1 and 0.9, there are two middle points, 0 and 0.5, and which is the *average*? The answer is both can be right. If we calculate the average from the point of view at, let's say, 0.45, the average is 0.5 because distances to two points from 0.45 are -0.35 and 0.45, respectively. But from the point of view at 0.1, the average becomes 0, because distances to two points are 0 and -0.2. Then, an average in view of one value can be different from those in view of other values. This means that we need an anchor value to find an average, and that averages are relative. In Eq. (4), each node i is using its state value ϕ_i as the anchor, and is finding its own average at each time step. But there still exists an

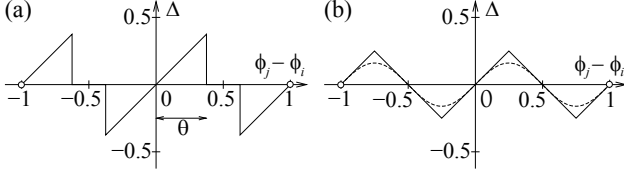


FIG. 3. The function $\Delta(\phi_j - \phi_i)$ representing the difference between ϕ_j and ϕ_i . (a) Rule A with a threshold θ in Eq. (5). (b) Rule B in Eq. (6). The dashed line is $\sin[2\pi(\phi_j - \phi_i)]/(2\pi)$, as in the Kuramoto model.

ambiguity when distances between two values are exactly 0.5; for example, if the anchor value is 0 and a value is at 0.5, the distance can be either 0.5 or -0.5. But, if we set $\Delta(x) = 0$ at $x = \pm 0.5$, this ambiguity can be avoided. In addition, $\Delta(x)$ should be periodic [$\Delta(x \pm 1) = \Delta(x)$ for $-1 < x < 1$], because x is a periodic variable.

One way to define $\Delta(x)$ is as below [see Fig. 3(a)],

$$\Delta_A(x) = \begin{cases} x+1 & (\text{if } x < -1+\theta), \\ 0 & (\text{if } -1+\theta \leq x \leq -\theta), \\ x & (\text{if } -\theta < x < \theta), \\ 0 & (\text{if } \theta \leq x \leq 1-\theta), \\ x-1 & (\text{if } 1-\theta < x), \end{cases} \quad (5)$$

where θ is the threshold for interaction ($0 < \theta \leq 0.5$), and we will call this rule as Rule A. Thresholds have been used in models of *bounded confidence*[6–8, 29] in opinion dynamics, and can influence the dynamics greatly. In this work, we set $\theta = 0.5$, the maximum value, for simplicity. Now we can find the next value of ϕ_i unambiguously. What happens if we repeatedly find averages using Eq. (4) for a given initial condition for N nodes? Values will eventually converge to a fixed point in the state space, where the average among neighbors for each node is the same as the state value of itself. This rule has been used in previous works by the author[11, 15]. Note that, due to the discontinuities in $\Delta_A(x)$, a slight change of the state for one node can induce abrupt changes of other nodes at the next time step.

We can avoid these discontinuities, and introduce another form of $\Delta(x)$ by modifying Eq. (5) as shown in the next equation [see Fig. 3(b)],

$$\Delta_B(x) = \begin{cases} x+1 & (\text{if } x \leq -0.75), \\ -x-0.5 & (\text{if } -0.75 < x \leq -0.25), \\ x & (\text{if } -0.25 < x \leq 0.25), \\ -x+0.5 & (\text{if } 0.25 < x \leq 0.75), \\ x-1 & (\text{if } 0.75 < x), \end{cases} \quad (6)$$

and we will call this rule as Rule B. This rule resembles the interaction term in the Kuramoto model[18], and we can use another form of $\Delta(x)$ as below,

$$\Delta_K(x) = \frac{1}{2\pi} \sin(2\pi x), \quad (7)$$

where results will be similar to those from Rule B (will not be shown here). Under both rules, A and B, each

node updates its state with the *averaged* value especially when $|\phi_j - \phi_i| < 0.25$, and we can call both of them as the rules of repeated averaging of periodic variables. One difference is that $|\Delta_B|$ decreases as $|x|$ increases from 0.25 to 0.5 unlike Rule A, making the meaning of averaging a little different. This can be interpreted as another representation of the concept of bounded confidence, because the influence between nodes starts to decrease when the difference is greater than 0.25. Later we will see that this fact can also play an important role for stability of some fixed points.

We know that the *converged* states (or *consensus* if we interpret ϕ_i as an opinion of node i , or *synchronized* states if we interpret ϕ_i as a phase of one of identical oscillators), where all nodes have the same value, are always stable fixed points under both rules. But there are other kinds of fixed points, and one type of fixed points can be represented by

$$\phi_i = \phi_0 \pm i \frac{n}{N} \pmod{1} \quad \text{for } 0 \leq i < N, \quad (8)$$

where n is a positive integer (when $n = 0$, it becomes the converged state). We can call these states as period- n states, and n is a period number. If ‘+’ (‘-’) is used in Eq. (8), ϕ_i increases (decreases) with the same rate as i increases. Here we look at the stabilities of these period- n states for small N ’s.

1. When $N = 3$, $E = 0$ and $d = 1$ (fully connected), period-1 states [e.g., $(0, \frac{1}{3}, \frac{2}{3})$] are fixed points under both rules; but they are stable under Rule A, while unstable under Rule B, because, if the difference between any pair of influencing nodes is greater than or equal to 0.25, where the $\Delta_B'(x)$ changes the sign, the state cannot be stable under Rule B.
2. When $N = 4$, $E = 0$ and $d = 1$, period-1 states [e.g., $(0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4})$] are fixed points under both rules; but they are stable under Rule A, but unstable under Rule B.
3. When $N = 4$, $E = 2$ and $d = 1$ (fully connected), period-1 states are unstable fixed points under both rules. They are unstable under Rule A, because, if the difference between any pair of influencing nodes is equal to 0.5, where $\Delta_A(x)$ changes the sign when $\theta = 0.5$, the state cannot be stable. No period- n state ($n > 0$) can be stable under Rule B when the network is fully connected, because there always exist a pair of nodes with the difference greater than 0.25.
4. When $N = 5$, $E = 0$ and $d = 1$, period-1 states are stable fixed points under both rules.
5. When $N = 5$, $E = 5$ and $d = 1$ (fully connected), period-1 states are fixed points under both rules; but they are stable under Rule A, while unstable under Rule B.

6. When $N = 10$, $E = 0$ and $d = 1$, both period-1 and period-2 states are stable fixed points under both rules.
7. When $N = 10$, $E = 0$ and $d = 2$, period-1 states are stable fixed point under both rules; but period-2 states are stable fixed points under Rule A, while become unstable under Rule B, because there exist pairs of nodes with the differences at 0.4.

We can easily see that, for fully-connected networks, converged states are the only stable fixed points under Rule B, while other stable fixed points can exist under Rule A. On the other hand, when $E = 0$, there exist stable fixed points other than converged states under both rules, but some stable fixed points under Rule A are unstable under Rule B. From above examples, it is easy to find that, when $n \geq \frac{N}{2}$ (Rule A) or $n \geq \frac{N}{4d}$ (Rule B), period- n states when $E = 0$ become unstable. In the next section, we will find properties of all possible stable fixed points in more detail using numerical simulations, when $N = 1000$ and $d \ll N$ with various e values.

III. NUMERICAL RESULTS

Once an initial state is given, the state will converge to a fixed point eventually. Here we will categorize fixed points, and observe what types of fixed points are reached for given e under both Rule A and Rule B. Then, using the Monte-Carlo simulations, we will also estimate ratios of initial states, for which a certain types of fixed points will be reached. In other words, we will estimate average volumes of the basins of attraction for some types of fixed points, which form attractors. For example, if a network structure is given, the set of all converged states can be viewed as an attractor for the given dynamical rule, and states in a certain portion of the state space will reach converged states as time increases. If we choose N_r random initial conditions from uniformly distributed states and N_0 of them reach converged states, N_0/N_r is the estimated volume of the basin of attraction for converged states (the volume of the whole state space, N -torus, is 1). If $N_0/N_r \simeq 0$, the volume of the basin of attraction is extremely small, and the converged state is unlikely to be reached spontaneously in the system even though the converged states are stable fixed points, unless initial conditions are chosen deliberately. On the other hand, if $N_0/N_r \simeq 1$, the basin of attraction for converged states can be close to the whole state space.

In our simulations, if E is either 0 or $E_{N,d}$ ($e = 0$ or 1), there is only one possible network configuration. But if E is between 0 and $E_{N,d}$ ($0 < e < 1$), there are many possible configurations; hence we pick both a network configuration for given E and an initial condition, randomly out of uniformly distributed choices, for each run. Then, N_0/N_r will represent estimates of the *average* volumes of those basins of attraction for converged states over all possible network configurations. The same

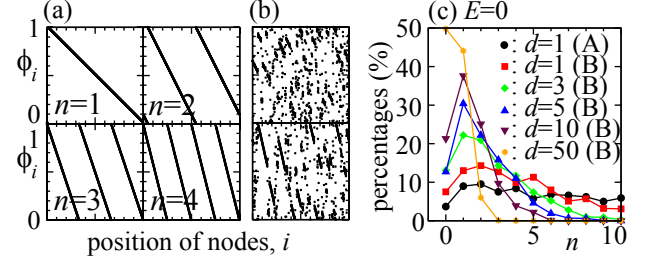


FIG. 4. (Color online) For $N = 1000$ and $E = 0$, we observe (a) examples of period- n states; (b) examples of typical fixed points when $d > 1$ for Rule A; and (c) ratios of period- n states out of 1000 random initial conditions under both Rule A ($d = 1$) and Rule B (various d 's).

method can be used on period- n states, too. We will still use $N_r = 1000$ for all our simulations.

First, we find fixed points when $N = 1000$ and $E = 0$ (see Fig. 4). Typical fixed points here are period- n states defined in Eq. (8). For $d = 1$ under Rule A and any d under Rule B, period- n states are only fixed points found [see Fig. 4(a)], but for $d > 1$ under Rule A, other types of fixed points exist as shown in Fig. 4(b) [meaning that Eq. (8) is not the only solution that satisfy $\forall i, \phi_i(t+1) = \phi_i(t)$]. Actually they dominate because no period- n state was reached from 1000 randomly given initial conditions when $k > 1$ under Rule A. These states have small *locally-converged*[30] segments with scattered values. In Fig. 4(c), we observe percentages of period- n states, including converged states ($n = 0$), out of 1000 random initial conditions. The higher n , the less likely they will be reached from random initial conditions in most cases. Under Rule B, as d increases, the highest n that can be found in fixed points decreases, and the volume of the basin of attraction for converged states increases; for example, when $d = 50$, about 50% of initial conditions reach converged states. For $d = 2$ and 3 under Rule A, percentages of period- n states including converged states are all zero in our simulations.

Next, we observe examples of fixed points under both rules as we vary e from 0 to 1 for the case of $N = 1000$ and $d = 3$ in Fig. 5. If we look at fixed points under Rule A, all fixed points are similar to states in Fig. 4(b) when E is less than about 1000 as shown in Figs. 5(a)-5(c). As E increases further, gathered states like the one shown in Fig. 5(d) start to appear, while diversely spread states still exist as in Fig. 5(e) [both are at $E = 3162$ ($e \simeq 6.5 \times 10^{-3}$)]. Converged states starts to appear when $E \simeq 4000$, and starts to dominate at $E \simeq 20\,000$ with about 97% of all fixed points. At this point, another types of fixed points called *grouping* states start to appear. In these states, nodes are divided into separated groups of nodes with clustered state values, and they can be called as m -group states, where m stands for the number of groups. For the case of $\theta = 0.5$ (Rule A), even-numbered groups are not reached[31], while, under Rule B, grouping

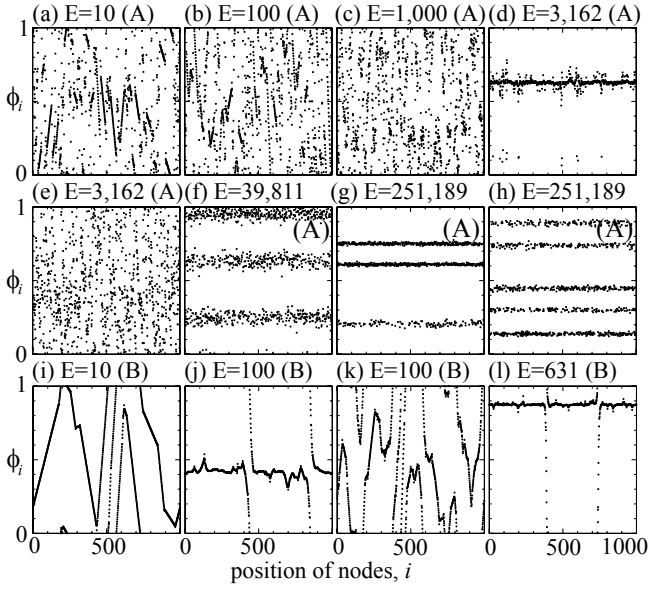


FIG. 5. For $N = 1000$ and $d = 3$, we show examples of stable fixed points for various E values numerically. Network configurations and initial conditions are randomly picked for given E 's. (a)-(h) Under Rule A, (i)-(l) Under Rule B.

states are unstable, and will not be reached. When $m = 1$, states are just converged states, so 3-group states are the ones that appear mostly [see Figs. 5(f) and 5(g)]. Widths of state values for groups tend to decrease as E increases further, reaching 0 at $E = E_{N,d}$, while grouping states with $m \geq 5$ also starts to appear at $E \simeq 120\,000$ [see Fig. 5(h)].

Under Rule B, fixed points look a little different. First of all, stable fixed points are mostly locally converged throughout the whole structure, showing piecewise-linear shapes in the scatter plots in Figs. 5(i), 5(j), and 5(k) when $E = 10$ and 100. In these states, some can be gathered as in Fig. 5(j) and others can be spread as in Fig. 5(k), even at the same E value. Under Rule B, converged states starts to appear at $E \simeq 500$, and reach 100% at $E \simeq 1200$; while gathered states as in Fig. 5(l) at $E = 631$ and converged states coexist in this region. As E further increases, all states will lead to converged states.

If we categorize fixed points based on our numerical experiments so far, they are as below.

1. Converged states: All nodes have the same state value in these states, and they are stable in all cases even though the volume of the basin of attraction for these states can be almost zero when $e \ll 1$. All converged states satisfy

$$\phi_0 = \phi_1 = \phi_2 = \dots = \phi_{N-1}. \quad (9)$$

A set of all possible converged states can be viewed as an attractor, and the volume of the basin of attraction will be estimated numerically.

2. Periodic states: In these states, ϕ_i 's increase (or decrease) linearly as i increases along the structure with n cycles, and they are called as period- n states, where a positive integer n is in the range of $0 < n < \frac{N}{2}$ (Rule A) or $0 < n < \frac{N}{4k}$ (Rule B). We exclude $n = 0$, because period-0 states are converged state. These states are stable fixed points when $E = 0$ (when there is no long-range link). An equation for the period- n states is given in Eq. (8). For each n , there are two attractors, one with the set of increasing period- n states, and the other with the set of decreasing period- n states. Here we will estimate the sum of volumes of basins of attraction for these two attractors for each n .
3. Grouping states: Nodes can be divided into separated groups of nodes with clustered state values, and they can be called as m -group states, where m stands for the number of groups. We exclude $m = 1$, because 1-group states are converged states. Grouping states can be found only when $e \simeq 1$ under Rule A. If $e = 1$, where the network is fully connected, each group will consist of nodes with the same value. Groups don't have to be equally distanced; rather it depends on sizes of groups as shown in Fig. 5(g).
4. Other states: There can be fixed points that don't belong to above categories, and nodes look mostly scattered in the scatter plots. We can find some fixed points when N is small. For example, in Fig. 1(b), cases with $N = 4$ and 5 are shown, and we can find some fixed points under Rule A. For the case of $N = 4$, $E = 1$ and $k = 1$, $(0, \frac{1}{8}, \frac{1}{2}, \frac{7}{8})$ is a fixed point; for the case of $N = 5$, $E = 1$ and $k = 1$, $(0, \frac{1}{11}, \frac{4}{11}, \frac{7}{11}, \frac{10}{11})$ is a fixed point; and for the case of $N = 5$, $E = 2$ and $k = 1$, $(0, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3})$ is a fixed point. For $N \gg 1$ and $E \gg 1$, these fixed points can be highly complicated (see Fig. 5). If we categorize these states further, there can be two different types.

- Gathered states: As shown in Figs. 5(d) and 5(j), state variables are gathered near one value, even though some states can be far from this value.
- Spread states: As shown in Figs. 5(e) and 5(k), state variables are spread well between 0 and 1, even though we can find some patterns in the scatter plots.

We can elaborate more on two types of *other* states mentioned above. Under Rule A, when E is less than about 5000, the whole spectrum of states between these two types of states can be reached. But for the approximate range of $5 \times 10^3 < E < 1.5 \times 10^4$ ($10^{-2} < e < 3 \times 10^{-2}$), they are clearly separated (see Fig. 6). Under Rule B, though, states are locally converged throughout the whole structure; therefore states are represented by

an one-dimensional curves in scatter plots. For low E 's, these curves are piecewise linear, and points where linear segments meet are the locations where long-range connections branch off. If E is greater, the numbers of these branching points increase, too, and the curves become more complicated. As in Rule A, gathered and spread states can coexist for the cases with the same E .

To see how nodes are spread for these fixed points, we introduce a quantity s . First, we need an anchor point because we learned in the previous section that an anchor point is necessary to find the average value of ϕ_i 's. Here we set the anchor point ϕ_s as the point where the most nodes are gathered state-wise; in other words, if we find the probability distribution function, $P(\phi)$, the anchor point ϕ_s is the point where $P(\phi_s)$ is the maximum for a given fixed point. Second, we can find the average of all ϕ_i 's, $\langle \phi \rangle_s$, viewed from ϕ_s , using $\Delta_A(x)$ with $\theta = 0.5$,

$$\langle \phi \rangle_s = \phi_s + \frac{1}{N} \sum_{j=0}^{N-1} \Delta_A(\phi_j - \phi_s) \pmod{1}. \quad (10)$$

Finally, s is defined as

$$s \equiv \sqrt{\frac{12}{N} \sum_{j=0}^{N-1} [\Delta_A(\phi_j - \langle \phi \rangle_s)]^2}. \quad (11)$$

This is the standard deviation of node states viewed from ϕ_s [32]. For converged states, s should be 0, while, for uniformly distributed states like period- n states, s is 1. For other states, if it is close to 0, the state is gathered near one value; and if it is close to 1, the state is diversely spread. (Note that 1 is not the maximum possible value.) Using this index, we can see distributions of various fixed points in more detail.

In Fig. 6, we observe values of s using Monte-Carlo simulations, which show us what types of fixed points are more likely to appear spontaneously. Under Rule A [see Fig. 6(a)], all fixed points are spread states when E is less than about 1000 ($e \simeq 2 \times 10^{-3}$). But a transition occurs in the approximate range of $10^3 < E < 2 \times 10^4$ ($2 \times 10^{-3} < e < 4 \times 10^{-2}$), and gathered states start to appear. At $E \simeq 4 \times 10^3$ ($e \simeq 8 \times 10^{-3}$), converged states start to appear, too. All three kinds of fixed points coexist until $E \simeq 2 \times 10^4$ ($e \simeq 4 \times 10^{-2}$), where converged states are dominant (about 97%) and spread states no longer exist. At this point, grouping states start to appear, and 3-group states are the ones that appear mostly. For $2 \times 10^4 < E < 5 \times 10^4$ ($4 \times 10^{-2} < e < 10^{-1}$), converged states dominate and reach about 99%. Gathered and spread states have all disappeared, and grouping states start to increase as E further increases. Grouping states with $m \geq 5$ also starts to appear at $E \simeq 1.2 \times 10^5$ ($e \simeq 2.4 \times 10^{-1}$). Grouping states are in the top-right corner of Fig. 6(a).

Under Rule B, gathered and spread states also appear when E is small, even though states are mostly one-dimensional in scatter plots as seen in Fig. 5. Converged

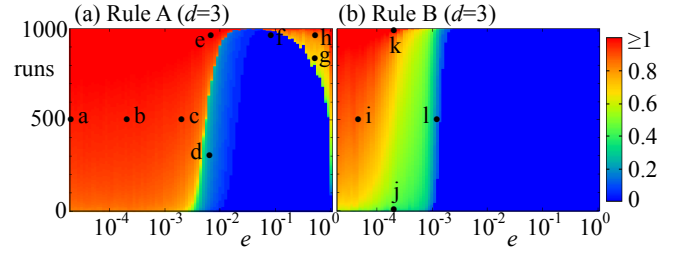


FIG. 6. (Color online) We observe s for 1000 runs for various E values ($E \geq 10$) when $N = 1000$ and $d = 3$. For given E , both positions of long-range links and an initial condition were chosen randomly for each run. Each pixel represents a run, and a column of pixels, representing runs for given E , are ordered, so that the run with the lowest s is in the bottom. Blue regions represent converged states ($s = 0$), and red regions represent quite diverse states ($s \geq 1$). (a) Values of s under Rule A. (b) Values of s under Rule B. There are 12 dots in graphs, indexed by alphabets, and they correspond to the states depicted in Fig. 5.

states start to appear at $E \simeq 500$ ($e \simeq 10^{-3}$), and their distributions reach 100% at $E \simeq 1200$ ($e \simeq 2.4 \times 10^{-3}$). Note that more than 10% of initial conditions reach converged states when $E = 0$ as shown in Fig. 4(c). But those converged states disappear before E reaches 10. When E is greater than 1200, only converged states are observed, meaning that the volume of the basin of attraction for converged states is the volume of the whole state space. There are no grouping states here as discussed earlier. One interesting point is that, for the region of e for small-world behavior (when e is less than about 10^{-3}), reached fixed points are all diverse states without any converged state.

Now we turn our attention to the case of $d = 0$, where all links are randomly attached. It is easy to see that a necessary condition for a network to have fixed points other than converged states is for the network to be cyclic. For acyclic networks, only converged states are fixed points. In our model for $d > 0$, short-range links guarantee that these networks have at least a cycle, and we observed many types of fixed points so far. What happens if there is no prearranged short-range links for the case of $d = 0$? First, there can be more than one component, a group of connected nodes, for low E 's; hence we only consider states of nodes for the largest component when calculating s and $P(\phi)$, the probability distribution function of ϕ_i 's for a given fixed point, since we are interested in looking at how connected nodes influence each other. Second, the chance of the largest component of the random network to be acyclic is not negligible when E is low, but, as E increases, that chance becomes extremely small. But, for higher E 's, we expect to see similar results as those for the cases of $d > 0$, because influences of short-range links become negligible when $E \gg dN$.

In Fig. 7, we look at some examples of fixed points for the case of $N = 1000$ and $d = 0$, and observe behavior of

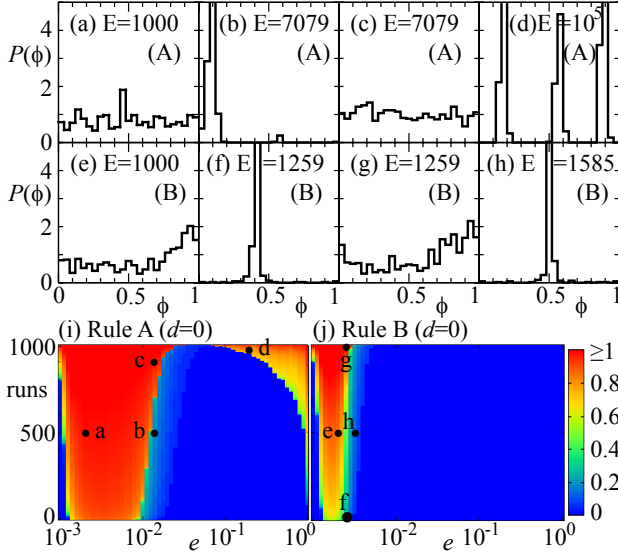


FIG. 7. (Color online) We observe s for 1000 runs for various E values ($E \geq 500$) when $N = 1000$ and $d = 0$. (a)-(d) Distributions of states $P(\phi)$ under Rule A. (e)-(h) Distributions of states $P(\phi)$ under Rule B. (i) Values of s under Rule A. (j) Values of s under Rule B. There are 8 dots in graphs, indexed by alphabets, and they correspond to the states in this figure.

s as in Figs. 5 and 6. Since scatter plots of ϕ_i versus i are no longer meaningful, we use the probability distribution function $P(\phi)$ to represent each fixed point. In Figs. 7(a)-7(h), some fixed points under both rules are shown. We can find all types of fixed points except the periodic states. We can also observe 2-group states under Rule A here, unlike the cases of $d > 0$ [for 2-group states, a group usually consists of only few nodes as shown in Fig. 7(b)]. In Figs. 7(i) and 7(j), we observe s values for 1000 runs for each E under both rules. When E is less than about 600 ($e < 1.2 \times 10^{-3}$), converged states are dominant as stable fixed points, but, as E increases further, diverse states appear and later disappear with the same kind of transition from diverse states to converged states as we have already observed for cases of $d = 3$ in Fig. 6.

One important dynamic property on networks we have been using is this transition from diverse states to converged states as the number of links E increases. (Under Rule A, grouping states appear after this transition.) Then, we can ask when this transition occurs. One way to quantify the answer is finding the point at which the ratio of converged states out of all runs becomes exactly a half, and we will use the average connectivity, instead of E , to represent this value: k_c [note that $k = 2(d + E/N)$]. In Fig. 8(a), we show ratios of converged states out of 1000 runs versus k for four cases we have used at $N = 1000$: $d = 0, 3$ (Rule A), and $d = 0, 3$ (Rule B). Under Rule A, two curves match very well, and k_c is about 27; while under Rule B, two curves don't match because the numbers of nodes and links for the largest component are much

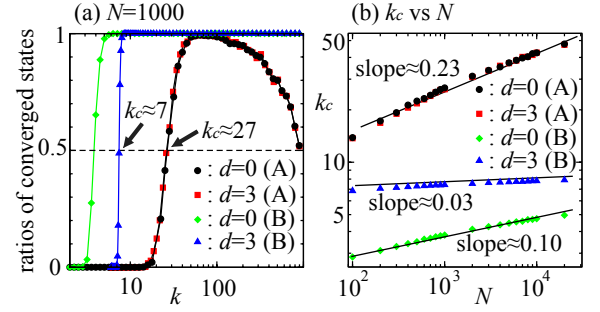


FIG. 8. (Color online) We observe values of E at which converged states start to prevail under both rules. (a) Ratios of converged states out of 1000 runs versus k , the average connectivity, when $d = 0$ and 3 under both rules. Values of k at which ratios reach 0.5 will be called as k_c . (b) k_c versus N for four cases. Slopes are exponents, ξ 's, in $k_c \propto N^\xi$.

smaller than N and E at $k \simeq k_c$ when $d = 0$. (For the case of $d = 0$ under Rule A, on the other hand, the possibility of having more than one component is extremely low at $k \simeq k_c$.) Finally, in Fig. 8(b), we observe how k_c changes as N increases. Overall dynamic behavior will not be much different, but how k_c varies with N ? Numerical results show that k_c fit well with N^ξ , and exponents ξ 's for four cases are also found. Exponents don't seem to depend on d under Rule A, and $\xi \simeq 0.23$; while under Rule B, $\xi \simeq 0.10$ ($d = 0$) and $\xi \simeq 0.03$ ($d = 3$). Results suggest that more long-range links are needed to get converged states under Rule A, and k_c increases faster with N , compared to results under Rule B. It is also interesting that, under Rule B ($d = 3$), k_c doesn't increase much with N (for example, $k_c \simeq 7.4$ at $N = 10^3$ and $k_c \simeq 7.8$ at $N = 10^4$).

IV. DISCUSSIONS

Using a simple network model, we have found fixed points for periodic state variables using two dynamic rules of repeated averaging, Rule A and Rule B. This model has a parameter e ($0 \leq e \leq 1$), representing the number of long-range links, and stable fixed points and their likelihood of appearing spontaneously from random initial conditions have changed as e varies. We categorized fixed points as converged states, periodic states, grouping states, and other states (gathered and spread). Grouping states are only reached under Rule A, while other types of fixed points have been found under both rules. As the number of long-range links increases, the transition from diverse states to converged states occurred in all cases, and we represented the point of transition using k_c , the average connectivity when the ratio of converged states becomes exactly 0.5. Numerically found k_c 's fit well with N^ξ , and exponents, ξ 's, have been found. For Rule A, we need more long-range connections (higher k_c) to get converged states, and the exponent ξ is

greater, compared to results under Rule B. In addition, diverse states can reappear in the form of grouping states under Rule A, when the network is close to being fully connected.

The same dynamic rule (Rule A) has been used in previous works[11, 15] for dynamic interaction networks with random-walking nodes in two-dimensional lattices. In these networks, short-range links constantly change as nodes move randomly in the lattice, thereby giving stochasticity to the model. Unlike the static network used here, converged states are the only stable fixed points, and periodic and grouping states are found to be metastable with finite lifetimes. For dynamic cases, periodic states are metastable when $0 < e \ll 1$, while they can be fixed points when $e = 0$ here. In addition, fixed points like gathered and spread states cannot be stable in dynamic networks, because these states are only stable specifically for given network configurations.

The motivation for this work is to answer the questions of how the diversities exist in reality even though many mechanisms of assimilation are at work, and how these diversities disappear if we increase long-range interactions. For example, if we look at human cultures, they are still

quite diverse, but neighboring cultures differ only slightly because neighbors usually interact and assimilate. One can use interaction networks to describe these systems, and argue that the prevalence of short-range interactions are the main cause of these kinds of diversities. Our model showed that, if there are only short-range links, periodic states, which are locally converged and yet diverse, can be stable fixed points. The point of transition from diversity to convergence we found in this model can be used as a guide when we roughly determine how many long-range links are necessary to make a group of nodes converge to one value. Even though the reality is more complex, we hope this simple model can give us insight when we try to understand phenomena of sustained diversities and their breakdown.

ACKNOWLEDGMENTS

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 - [31] For even-numbered groups, there exist two groups that have the difference of states at 0.5, and that is why these states are mostly unstable.
 - [32] To make s for uniformly distributed states normalized, the standard deviation has been multiplied by $\sqrt{12}$.